

**18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)**  
**More on the zeroes of  $\zeta$**

In this unit, we derive some results about the location of the zeroes of the Riemann zeta function, including a small zero-free region inside the critical strip.

## 1 Order of an entire function

For  $\alpha > 0$ , an entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is said to have *order*  $\leq \alpha$  if for all  $\beta > \alpha$ ,

$$f(z) = O(\exp |z|^\beta) \quad (|z| \rightarrow \infty).$$

We say  $f$  has order  $\alpha$  if it has order  $\leq \alpha$  but not order  $\leq \beta$  for any  $\beta < \alpha$ .

**Lemma 1.** *The function*

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

*satisfies*

$$|\xi(s)| < \exp(C|s| \log |s|) \quad (|s| \rightarrow \infty),$$

*and so is of order  $\leq 1$ . (An analogue is true for  $L$ -functions, but that is too easy even to give as an exercise.)*

*Proof.* By the functional equation  $\xi(s) = \xi(1-s)$ , it suffices to check for  $|\operatorname{Re}(s)| \geq 1/2$ , in which case

$$\left| \frac{1}{2}s(s-1)\pi^{-s/2} \right| < \exp(C_1|s|)$$

$$|\Gamma(s/2)| < \exp(C_2|s| \log |s|)$$

(see exercises for the second estimate). For  $\zeta$ , we use the integral representation from the first lecture:

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty (x - \lfloor x \rfloor) x^{-s-1} dx \quad (\operatorname{Re}(s) > 0).$$

For  $\operatorname{Re}(s) \geq 1/2$ , the integral is bounded, so  $|\zeta(s)| < C_3|s|$ . This yields the claim.  $\square$

There is a rich theory of integral functions of finite order due to Hadamard (which I believe was introduced originally for the very purpose of studying  $\zeta$ ). The basic idea is to generalize the fact that a polynomial can be written as a product of linear factors (the Fundamental Theorem of Algebra), to write an entire function as a product of one factor for each zero times an exponential.

To do this, one must first control the number of zeroes of  $f$  in a disc. There is no harm in assuming that  $f(0) \neq 0$ , since otherwise we just divide by a suitable power of  $z$ . Then recall the following fact from complex analysis.

**Theorem 2** (Jensen's formula). *If  $f(0) \neq 0$  and  $f$  has no zeroes on the circle  $|z| = R$ , then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta = \log |f(0)| + \sum_{\rho} (\log R - \log |\rho|),$$

where  $\rho$  runs over the zeroes of  $f$  in the disc  $|z| < R$  counted with multiplicity.

*Proof.* Write  $f(z) = (z - \rho_1) \cdots (z - \rho_n)g(z)$ , where  $g$  is nonzero on the disc  $|z| \leq R$ , and check the equality for each factor individually. For  $z - \rho_i$ , this is an easy exercise; for  $g$ , apply the Cauchy residue formula to the contour integral  $\int \log(g(z)) \frac{dz}{z}$  around the circle  $|z| = R$ , then take real parts.  $\square$

The right side is also

$$\log |f(0)| + \int_0^R \#\{\rho : |\rho| < r\} \frac{dr}{r}.$$

If  $\log |f(z)| < r(|z|)$  for some function  $r$ , then the left side of Jensen's formula is bounded by  $2r(R)$ , whereas the right side is at least

$$\log |f(0)| + \log(2) \#\{\rho : |\rho| \leq R/2\}.$$

Consequently, if  $r(R) = O(R^\alpha)$ , then the number of roots of  $f$  in the disc  $|\rho| \leq R$  is also  $O(R^\alpha)$ . Similarly, the fact that  $\log |\xi(s)| = O(|s| \log |s|)$  implies that the number of zeroes of  $\zeta$  with  $|\operatorname{Im}(s)| \leq T$  is  $O(T \log T)$ , which I claimed without proof in the previous unit.

Now let  $f$  be entire of order  $\leq 1$ . Let  $\rho_1, \rho_2, \dots$  be the zeroes of  $f$  sorted so that  $|\rho_1| \leq |\rho_2| \leq \dots$ , and put

$$h(z) = \prod_{n=1}^{\infty} (1 - z/\rho_n) e^{z/\rho_n}$$

Note that this converges uniformly on any disc, because the multiplicand is

$$1 + \frac{1}{2} \left( \frac{z}{\rho_n} \right)^2 + O \left( \left( \frac{z}{\rho_n} \right)^3 \right)$$

and the fact that the number of roots of norm  $\leq R$  is  $O(R^{1+\epsilon})$  implies that  $\sum 1/\rho_n^2$  converges (by partial summation). By a somewhat intricate argument (see Davenport §11 or Ahlfors), it can be shown that  $f/h$  is also of order  $\leq 1$ . Since  $f/h$  has no zeroes, the function  $g(z) = \log(f(z)/h(z))$  is entire and satisfies  $|g(z)| = O(|z|^{1+\epsilon})$ . Consequently,

$$g_2(z) = \frac{g(z) - g(0) - g'(0)z}{z^2}$$

is entire and bounded, hence constant by Liouville's theorem. This yields the following.

**Theorem 3** (Hadamard). *Let  $f(z)$  be an entire function of order  $\leq 1$ . Then*

$$f(z) = e^{A+Bz} \prod_{n=1}^{\infty} (1 - z/\rho_n) e^{z/\rho_n}$$

for some constants  $A, B$ .

## 2 A zero-free region for $\zeta$

We now use the product representation for  $\xi$  to obtain a zero-free region for  $\zeta$ . The idea (due to de la Vallée Poussin (1899)) is to squeeze a bit of extra information out of the proof we used for nonvanishing on the line  $\text{Re}(s) = 1$ . One way to phrase that argument: since

$$\text{Re}(\log(\zeta(s))) = \sum_p \sum_{n=1}^{\infty} \frac{1}{n} \cos(\text{Im}(s) \log p^n) p^{-n \text{Re}(s)}$$

and

$$3 + 4 \cos \theta + \cos 2\theta \geq 0,$$

we have

$$3 \text{Re}(\log \zeta(\sigma)) + 4 \text{Re}(\log \zeta(\sigma + it)) + \text{Re}(\log \zeta(\sigma + 2it)) \geq 0 \quad (\sigma > 1, t \in \mathbb{R})$$

whereas if  $\zeta(1 + it)$  vanished, then the sum would tend to  $-\infty$  as  $\sigma \rightarrow 1^+$  (because  $4 > 3$ ).

We can apply the same argument with  $\log \zeta$  replaced by its negative derivative

$$-\text{Re} \zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-\text{Re}(s)} \cos(\text{Im}(s) \log n)$$

to obtain an analogous inequality

$$-3 \text{Re} \frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4 \text{Re} \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} - \text{Re} \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \geq 0 \quad (\sigma > 1, t \in \mathbb{R}). \quad (1)$$

Let's see how to use (1) to get some information about zeroes just past the line  $\text{Re}(s) = 1$ . We do this by bounding above each term on the left side of (1) for  $\sigma$  slightly bigger than 1. For starters, since  $\zeta$  has a simple pole at  $s = 1$ ,

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + *$$

where every  $*$  in this argument is a positive constant, but no two need be the same.

Applying Hadamard's theorem and taking a logarithmic derivative, we get

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

Adjusting to get rid of the gamma factors, we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s - 1} - B - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'((s + 1)/2)}{\Gamma((s + 1)/2)} - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

For  $1 \leq \operatorname{Re}(s) \leq 2$  and  $|\operatorname{Im}(s)| \geq 1$ , everything on the right side aside from the sum over  $\rho$  is dominated by  $*\log |\operatorname{Im}(s)|$ . Hence taking real parts, we obtain

$$-\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)} < *\log |\operatorname{Im}(s)| - \sum_{\rho} \operatorname{Re} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Since  $\operatorname{Re}(\rho) > 0$  and  $\operatorname{Re}(s-\rho) > 0$ , we also have  $\operatorname{Re}(1/\rho) > 0$  and  $\operatorname{Re}(1/(s-\rho)) > 0$ , so the sum over  $\rho$  is positive. Hence

$$-\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)} < *\log |\operatorname{Im}(s)|;$$

this is the estimate I'll use for  $s = \sigma + 2it$ .

Let  $t$  be the imaginary part of a zero  $\rho$  of  $\zeta$ ; I will bound  $-\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)}$  for  $s = \sigma + it$  by keeping only the summand corresponding to  $\rho$ . Namely, if  $\rho = \beta + it$ , then I get

$$-\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)} < *\log |t| - \frac{1}{\sigma - \beta}.$$

From (1), I now deduce

$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + *\log |t|.$$

For  $\sigma = 1 + * / (\log |t|)$ , I can deduce

$$\beta < 1 - \frac{*}{\log |t|}.$$

In other words:

**Theorem 4.** *There exists a constant  $c > 0$  such that there is no zero of  $\zeta$  in the region  $\operatorname{Re}(s) \geq 1 - c / \log \operatorname{Im}(s)$ ,  $\operatorname{Im}(s) \geq 1$ .*

By von Mangoldt's formula (presented in the previous unit, with proof still to follow), this yields a nontrivial error bound in the prime number theorem, namely

$$\pi(x) = \operatorname{li}(x) + O(x \exp(-c\sqrt{\log x}))$$

(exercise).

### 3 What about $L$ -functions?

The previous argument goes through more or less unchanged for  $L$ -functions. But there is a new complication: remember that we only looked at zeroes whose imaginary part was not too small. We took  $|\operatorname{Im}(s)| \geq 1$ , but the lower bound could have been any *fixed* positive constant.

The real issue is that while we can check once and for all that  $\zeta(s)$  has no zeroes on the real line, we cannot rule this out for  $L$ -functions. But  $L(s, \chi)$  could in principle have a real zero; such a hypothetical zero is called a *Siegel zero*. These can only occur for real nonprincipal characters.

# Exercises

1. Prove that  $1/\Gamma$  is entire of order  $\leq 1$ . Then prove that

$$\frac{1}{s\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} (1 + s/n)e^{-s/n} \quad (s \neq 0, -1, -2, \dots),$$

where  $\gamma$  is Euler's constant, by applying Hadamard's theorem.

2. Prove that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log(s) + O(|s|^{-1}) \quad (|s| \rightarrow \infty, \operatorname{Re}(s) \geq 1/2).$$

(Hint: use the previous exercise.)

3. Derive the estimate

$$|\Gamma(s/2)| < \exp(C_2 |s| \log |s|) \quad (\operatorname{Re}(s) \geq 1/2)$$

by first proving a suitably strong version of Stirling's formula, e.g.,

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}) \quad (|s| \rightarrow \infty, \operatorname{Re}(s) \geq 1/2).$$

4. Prove that a function of order  $\leq \alpha$  need not satisfy  $|f(z)| = O(\exp(|z|^\alpha))$ . (Hint: look at  $\zeta$  on the positive real axis.)
5. Find the constants  $A$  and  $B$  in the product representation for  $\xi$  given by Hadamard's theorem. Then deduce as a corollary that  $\frac{\xi'(0)}{\xi(0)} = \log 2\pi$ .
6. Use the zero-free region and von Mangoldt's formula to prove that for some  $c > 0$ ,

$$\pi(x) = \operatorname{li}(x) + O(x \exp(-c\sqrt{\log x})).$$

(By contrast, the leading term is  $x \exp(-\log \log x)$ .)